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# Torsional load transfer from a rigid shaft to an elastic plane with a slit

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**Abstract.** A plane elastostatic problem for an elastic wedge loaded by a concentrated moment at its apex provides an example of violation of the Saint-Venant principle for apex angles $2\alpha$ larger than $\pi$. Considering the problem for a truncated wedge, Neuber demonstrated the method of construction of an applicable solution for any apex angles in the range $\pi \leq 2\alpha \leq 2\pi$, despite the failure of the Saint-Venant principle. In the present paper the particularly important case of the truncated-wedge problem is examined. The truncated wedge degenerates into a slitted elastic plane, while a rigid circular shaft, acted upon by a torsional moment, is inserted into the plane. The analytical solution of the mixed boundary-value problem is obtained. Numerical results turn out to be in complete agreement with Neuber's results for the slitted elastic plane.

**Key words:** Carothers paradox, concentrated couple, force transfer, homogeneous solutions, slitted elastic plane, truncated wedge

## 1. Introduction

A two-dimensional problem for an elastic wedge occupying the domain $0 \leq r < \infty$, $-\alpha \leq \vartheta \leq \alpha$ in a polar-coordinate system $(r, \vartheta)$ with free surfaces $\vartheta = \pm\alpha$ and loaded by a concentrated moment $M$ at its apex was solved independently quite a long time ago by Fillunger [1], Carothers [2] and Inglis [3]. The distribution of normal ($\sigma_r$ and $\sigma_\vartheta$) and tangential ($\tau_{r\vartheta}$) stresses is given by

$$\sigma_r = -\frac{2M \sin 2\vartheta}{(2\alpha \cos 2\alpha - \sin 2\alpha)r^2}, \quad \sigma_\vartheta = 0, \quad \tau_{r\vartheta} = \frac{M(\cos 2\vartheta - \cos 2\alpha)}{(2\alpha \cos 2\alpha - \sin 2\alpha)r^2}. \tag{1}$$

Traditionally, this solution is called the 'Carothers solution'. It corresponds to the following Airy stress function (see [1–3])

$$\Phi = \frac{M(\sin 2\vartheta - 2\vartheta \cos 2\alpha)}{2(2\alpha \cos 2\alpha - \sin 2\alpha)}, \tag{2}$$

which is a certain solution of the fundamental equation of elasticity, specifically the homogeneous biharmonic equation.

The Carothers solution (1) satisfies the rough definition of the degenerate problem that can be formulated as follows: while body forces are ignored, the tractions vanish on both flanks of the unbounded wedge and produce a constant resultant moment $M$ (with zero resultant force) on any contour about the apex.

Fillunger [1] (see also [4]) was the first author who noticed that representation (1) has a curious behaviour at the single, physically significant, wedge angle: it tends to infinity at the apex angle $2\alpha^* = 257°24'$, whereas a magnitude $M$ of the moment remains finite. Later this pathological behaviour was called [5] the 'Carothers paradox'.

Sternberg and Koiter [5] considered the non-degenerate modified problem, in which the couple is replaced by a statically equivalent continuous load distributed on the flanks close to the apex. When attempting to ascertain whether the solution (1) of the degenerate problem is asymptotically approached by some solutions of non-degenerate problems, they arrived at the correct conclusion: the failure of the Saint-Venant principle occurs at apex angles in the range $2\alpha^* \leq 2\alpha < 2\pi$. Moreover, in this case the idealized notion of a concentrated couple applied to the wedge's apex is meanlingless. However, Sternberg and Koiter [5] did not provide any method of constructing an applicable solution.

The paper [5] gave rise to furious discussions and subsequently this fundamental problem in the theory of elasticity has been investigated by many researchers; for a comprehensive survey of the literature see [6]. Thus, the continuity of a stress distribution at $2\alpha = 2\alpha^*$ was demonstrated by Sonntag [7] from a physical-technical point of view by photoelastic experiments. While the major part of these studies has either a descriptive or formal mathematical character, including considerations of a larger class of similar phenomena, one should mention [8–10] and comparatively recent papers [11, 12], in particular. Considering the behaviour of the stress field in terms of a near-field geometrical effect, and a far-field stress interference effect at the free sides of a plane-strain elastic wedge, Stephen and Wang [12] are favourably noted by their in-depth study and the scope of their results.

A suitable engineering solution was constructed by Neuber [13]. He considered the problem in which the apex is located beyond the domain of the wedge and isolated by a circular cylinder of radius $\varepsilon$ which acts as 'force-transfer surface'. Trying to select external forces which are realizable in practice, Neuber [13] enunciated the principles of force transfer. These principles led to the points of the intrinsic significance referring to a character of a shearing stress distribution on the force transfer surface: the distribution is to have a constant direction and to vary moderately. Another important proposition in [13] concerns the selection of homogeneous solutions (the non-zero stress distributions in the wedge which satisfy zero boundary conditions at the flanks) for the truncated wedge. In order to realize the indicated principles according to Neuber [13], it is sufficient to select only two homogeneous solutions, so that one of them possesses a self-equilibrated peculiarity and another corresponds with the Carothers solution (see Section 5). By adding these solutions Neuber [13] obtained the 'exact' solution useful for practical purposes that remains continuous at any apex angle $2\alpha$ of the wedge.

The energy method proposed by Ulitko [14] is based on minimum principles of elasticity and two homogeneous solutions selected by Neuber [13]. It has the advantage that the constant which Neuber [13] specified with inequalities is determined uniquely.

The present paper addresses the analytical solution of the truncated-wedge mixed problem for the important particular case of a slitted elastic plane. In contrast to the problem posed by Neuber [13], the torsional moment is applied by means of a rigid circular shaft with given tangential displacements at its surface. Nevertheless, this statement agrees with Neuber's approach and it is sufficcently effective in technical aspects. The main goal of this paper is to verify Neuber's results for the wedge angle $2\alpha = 2\pi$ and thus the correctness of Neuber's approach. Proceeding from the method of homogeneous solutions, we reduce the problem to solving a singular infinite system of linear algebraic equations that can be brought to a regular form. An asymptotic analysis of the singular system's equations and a regular system's solution has been carried out. On the basis of a numerical solution of the regular infinite system normal $\sigma_r$ and shearing $\tau_{r\vartheta}$ stresses are plotted as functions of the angular coordinate $\vartheta$ on a

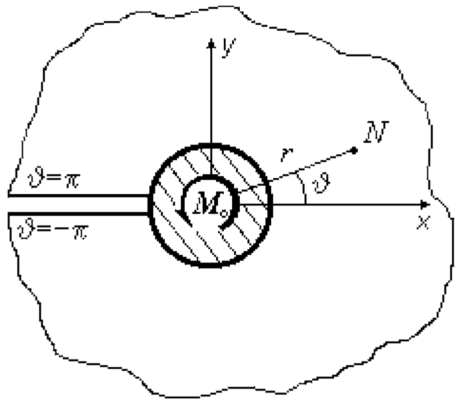


*Figure 1.* Slitted elastic plane joined with the rigid shaft.

joint area between the elastic plane and the rigid shaft. Finally, the graphs display somewhat unexpected results: these relations are found to be in complete qualitative agreement with Neuber's deductions.

## 2. Statement of the problem

A rigid shaft (cylinder) of radius $\varepsilon$ is fitted into an elastic plane with a semi-indefinite slit outside the circle of the same radius (see Figure 1). The shaft is rigidly joined with the plane across the contact zone $r = \varepsilon, -\pi \le \vartheta \le \pi$. A torsional moment $M_0$ is acting on the shaft which as a result rotates through a certain angle $\varphi_0$ counterclockwise. Therefore, the displacements in the contact zone are prescribed as

$$u_\vartheta = \varepsilon \cdot \varphi_0, \quad u_r = 0, \qquad r = \varepsilon, \quad -\pi \le \vartheta \le \pi, \tag{3}$$

and the sides of the slit are free of tractions

$$\sigma_\vartheta = 0, \quad \tau_{r\vartheta} = 0, \qquad \varepsilon \le r < \infty, \quad \vartheta = \pm\pi. \tag{4}$$

As is usual, we assume that the displacements vanish at infinity $r \to \infty$.

It is easy to see that the problem defined agrees with the approach by Neuber [13] and it is natural in its technical aspect. As is the case with Neuber's treatment, a truncated wedge degenerating into the slitted plane is considered. Likewise, the circular cylinder of radius $\varepsilon$ acts as the force-transfer surface. However, the force transfer itself should be discussed more accurately. The boundary conditions (3) at the surface $r = \varepsilon$ (the unclosed circle) are prescribed with the displacements of its points which are dependent on the angle $\varphi_0$ of rotation of the shaft[1]. Hence the solution of the problem will be a function of $\varphi_0$ and is not related to the torsional moment $M_0$ explicitly. In other words, it is a matter of displacements transfer rather than of force transfer. The foregoing formulation of the problem, in which a force condition (*i.e.*, the condition of the static equilibrium under the action of the moment) is replaced by the geometrical condition (3), generalizes to a certain extent Neuber's principles. It is obvious that there is a one-to-one correspondence between the angle of rotation $\varphi_0$ and the torsional moment $M_0$ acting on the shaft. Therefore, full qualitative agreement with Neuber [13] is

[1] Note, that in [13] the boundary conditions at $r = \varepsilon$ are not formulated explicitly.

expected and it is worth considering the force transfer but the discussions take the following order: first on the basis of the displacements specified on the force-transfer surface $r = \varepsilon$ the stressed condition is to be determined throughout the elastic plane, particularly in the contact zone $r = \varepsilon$, $-\pi \le \vartheta \le \pi$ and afterwards one can evaluate the moment $M_0$ that actually raised the specified displacements and hence the stressed condition on the whole system.

Completing this section, we note that by means of the formula

$$M_0 = -\int_{-\pi}^{\pi} \tau_{r\vartheta}|_{r=\varepsilon} \cdot \varepsilon^2 \mathrm{d}\vartheta \tag{5}$$

that follows from the integral equilibrium conditions for the shaft, one can evaluate the torsional moment which is to be applied to the shaft when it is rigidly joined to the elastic plane, so that it will rotate through the given angle $\varphi_0$.

## 3. Method of solution

### 3.1. Homogeneous solutions for the slitted plane

On the basis of the general solution of the antisymmetric problem for the plane elastic wedge (see Uflyand [15, pp. 124–128]) it is easy to derive the particular solution of the fundamental equations of plane elasticity in the form

$$\begin{aligned}
\mathrm{e}^t \frac{\sigma_r}{2G} &= -\left[\frac{s+3}{s-1} c(s) \sin(s+1)\vartheta + d(s) \sin(s-1)\vartheta\right] \mathrm{e}^{-st},\\
\mathrm{e}^t \frac{\sigma_\vartheta}{2G} &= \left[c(s) \sin(s+1)\vartheta + d(s) \sin(s-1)\vartheta\right] \mathrm{e}^{-st},\\
\mathrm{e}^t \frac{\tau_{r\vartheta}}{2G} &= \left[\frac{s+1}{s-1} c(s) \cos(s+1)\vartheta + d(s) \cos(s-1)\vartheta\right] \mathrm{e}^{-st},\\
\frac{u_r}{\varepsilon} &= \frac{1}{s}\left[\frac{3-4\nu+s}{s-1} c(s) \sin(s+1)\vartheta + d(s) \sin(s-1)\vartheta\right] \mathrm{e}^{-st},\\
\frac{u_\vartheta}{\varepsilon} &= \frac{1}{s}\left[\frac{3-4\nu-s}{s-1} c(s) \cos(s+1)\vartheta - d(s) \cos(s-1)\vartheta\right] \mathrm{e}^{-st},
\end{aligned} \tag{6}$$

where previously the polar coordinate $r$ was replaced by the new variable $t$ as follows

$$r = \varepsilon \cdot \mathrm{e}^t, \qquad \varepsilon \le r < \infty,\ 0 \le t < \infty. \tag{7}$$

So far the parameter $s$, which may take on complex values in general, is undefined, whereas two arbitrary functions $c(s)$, $d(s)$ of $s$ are to be determined from the boundary conditions. Let us choose values to the parameter $s$ to meet the boundary conditions (4).

The first condition in (4) provides at once the relation as $d(s) = -c(s)$. In order to satisfy the second one, $s$ is to be determined from

$$\cos \pi s = 0. \tag{8}$$

That is, $s_k = k - 1/2$, $k = 0, 1, \ldots, \infty$. Hereafter it is convenient to denote

$$\frac{c(s_k)}{s_k(s_k-1)} \quad \text{with} \quad c_k, \quad \textit{i.e.}, \quad c\Big(k-\frac{1}{2}\Big) = \Big(k-\frac{3}{2}\Big)\Big(k-\frac{1}{2}\Big)c_k.$$

Therefore, summing over all values of $k$, the following homogeneous solution has been obtained, *e.g.*, for displacements at the surface $r = \varepsilon$ $(t = 0)$

$$\begin{aligned} \left.\frac{u_r^{(1)}}{\varepsilon}\right|_{t=0} &= \sum_{k=0}^{\infty} c_k \left[\left((3-4\nu) + \left(k - \frac{1}{2}\right)\right) \sin(k + \frac{1}{2})\vartheta - \left(k - \frac{3}{2}\right) \sin(k - \frac{3}{2})\vartheta\right], \\ \left.\frac{u_\vartheta^{(1)}}{\varepsilon}\right|_{t=0} &= \sum_{k=0}^{\infty} c_k \left[\left((3-4\nu) - \left(k - \frac{1}{2}\right)\right) \cos(k + \frac{1}{2})\vartheta + \left(k - \frac{3}{2}\right) \cos(k - \frac{3}{2})\vartheta\right]. \end{aligned} \tag{9}$$

On the other hand, the condition $\tau_{r\vartheta}|_{\vartheta=\pm\pi} = 0$ in (4) provides at once the relation $c(s) = -\dfrac{s-1}{s+1}\, d(s)$. In a similar way, one can establish the following homogeneous solution for the displacements at the surface $r = \varepsilon$ $(t = 0)$

$$\begin{aligned} \left.\frac{u_r^{(2)}}{\varepsilon}\right|_{t=0} &= -\sum_{n=1}^{\infty} \left[\big((3-4\nu) + (n-1)\big) d_{n-1} - (n+2) d_{n+1}\right] \sin n\vartheta, \\ \left.\frac{u_\vartheta^{(2)}}{\varepsilon}\right|_{t=0} &= -2d_1 - \sum_{n=1}^{\infty} \left[\big((3-4\nu) - (n-1)\big) d_{n-1} + (n+2) d_{n+1}\right] \cos n\vartheta, \end{aligned} \tag{10}$$

with the notations

$$\frac{d(s_n)}{s_n(s_n+1)} = \frac{d(n)}{n(n+1)} = d_n,$$

where $s_n = n,\ n = 1, 2, \ldots, \infty$ has been determined from

$$\sin \pi s = 0 \tag{11}$$

to satisfy $\sigma_\vartheta|_{\vartheta=\pm\pi} = 0$ in conditions (4). In Equation (10) $d_0$ denotes an arbitrary constant that is associated with a (antisymmetric) rigid-body displacement in the $y$-direction.

Thus, we have constructed a set of ‘homogeneous solutions’ confirming to the equilibrium of an elastic plane with a semi-indefinite crack (slit). These solutions retain the sides of the crack free of tractions. The mentioned set consists of two independent subsets which are defined by the roots of Equations (8) and (11), respectively. An independent particular solution complies with each root $s_k$ of Equation (8) or $s_n$ of Equation (11). Due to the factor $e^{-st}$ the corresponding stress distributions decay exponentially with distance from the force-transfer surface $r = \varepsilon$ $(t = 0)$. All of them, except the distribution raised by the root $s_n = 1$, are self-equilibrated. See [12] for a detailed analysis of self-equilibrated load effects as against loads constituting the non-zero stress resultants.

Note that one can use the technique suggested by Lourje [16] and determine values of $s$ from the following eigenequation

$$(\sin 2s\alpha - s\ \sin 2\alpha)|_{\alpha=\pi} = 0,\ \textit{i.e.}, \sin 2s\pi = 0. \tag{12}$$

To obtain Equation (12) one should consider a system of two linear algebraic equations for the unknowns $c(s)$ and $d(s)$ which follows from the boundary conditions (4). By putting the determinant of the system equal to zero to guarantee the existence of a non-trivial solution, we immediately come to Equation (12). Nevertheless, it is easy to see that the homogeneous solutions remain the same.

### 3.2. Functional equations for the boundary value problem

In order to satisfy the boundary conditions (3) on the force transfer surface, the specified homogeneous solutions may be applied, since they produce no effect on the sides of the crack. As mentioned before, the solutions (9) and (10) are independent. Therefore, it is necessary for completeness to add them together. Hence, the boundary conditions (3) at $r = \varepsilon$ $(t = 0)$ take the form

$$\frac{u_r}{\varepsilon}\bigg|_{t=0} = \frac{u_r^{(1)} + u_r^{(2)}}{\varepsilon}\bigg|_{t=0} = 0, \qquad \frac{u_\vartheta}{\varepsilon}\bigg|_{t=0} = \frac{u_\vartheta^{(1)} + u_\vartheta^{(2)}}{\varepsilon}\bigg|_{t=0} = \varphi_0 = \text{const}. \tag{13}$$

Substitution of the expressions (9) and (10) in (13) leads to the functional equations for the boundary-value problem. They may be used to determine the unknowns $c_k$ and $d_n$.

It is essential to note that the completeness of the system of homogeneous solutions which we employed is an intricate fundamental problem of elasticity. However, it is worth mentioning that it is positively settled in statical and dynamical problems of elasticity (see [18]).

### 3.3. Infinite singular system

When satisfying the boundary conditions on the force-transfer surface $r = \varepsilon$ $(t = 0)$, natural difficulties inevitably arise. They are attributable to the fact that the system of homogeneous solutions is not orthogonal. However, we have the functions sin, cos of half-integer arguments in (9) and the functions sin, cos of integer arguments in (10). Therefore, one can employ the following formulae

$$\begin{aligned} &\int_0^\pi \sin\Big(k + \frac{1}{2}\Big)\vartheta \cdot \sin n\vartheta \,\mathrm{d}\vartheta = (-1)^{n+k+1} \frac{n}{n^2 - \left(k + \frac{1}{2}\right)^2}, \\ &\int_0^\pi \cos\Big(k + \frac{1}{2}\Big)\vartheta \cdot \cos n\vartheta \,\mathrm{d}\vartheta = (-1)^{n+k+1} \frac{k + \frac{1}{2}}{n^2 - \left(k + \frac{1}{2}\right)^2} \end{aligned} \tag{14}$$

to convert the functional equations (13). It is clear that similar conversions are impossible for apex angles $2\alpha$ differing from $2\pi$.

Thus, in view of Equations (14), we obtain from the conditions (13) the following infinite system of coupled equations

$$\left\{ \begin{aligned} &\big(2(1-2\nu) + n\big) d_{n-1} - (n+2) d_{n+1} = (-1)^n \cdot n \cdot \frac{2}{\pi} \times \\ &\times \sum_{k=0}^{\infty} (-1)^k c_k \left[ \frac{2(1-2\nu) + \left(k + \frac{1}{2}\right)}{\left(k + \frac{1}{2}\right)^2 - n^2} - \frac{\left(k - \frac{3}{2}\right)}{\left(k - \frac{3}{2}\right)^2 - n^2} \right], \\ &\big(4(1-\nu) - n\big) d_{n-1} + (n+2) d_{n+1} = (-1)^n \cdot n \cdot \frac{2}{\pi} \times \\ &\times \sum_{k=0}^{\infty} (-1)^k c_k \left[ \Big(k + \frac{1}{2}\Big) \frac{4(1-\nu) - \left(k + \frac{1}{2}\right)}{\left(k + \frac{1}{2}\right)^2 - n^2} + \frac{\left(k - \frac{3}{2}\right)^2}{\left(k - \frac{3}{2}\right)^2 - n^2} \right], \qquad n = 1, 2, \ldots, \end{aligned} \right. \tag{15}$$

in particular for $n = 0$:

$$2d_1 + \varphi_0 = -4(1-\nu) \cdot \frac{1}{\pi} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k c_k}{k+\frac{1}{2}}. \tag{16}$$

Now let us eliminate the terms $d_{n-1}$ and $d_{n+1}$ between Equations (15) and (16). With that end in mind one can solve the system of Equations (15) for $d_{n-1}$ and $d_{n+1}$. Then it should be accounted for that $d_{n-1}$ and $d_{n+1}$ are to be equal if one substitutes $n+2$ for $n$ in the expression obtained for $d_{n-1}$. Omitting the details of these tedious computations, we write immediately the results as follows:

$$\sum_{k=0}^{\infty} \widetilde{a}_{k0}\, X_k = 1\ ,\ n=0\ , \quad \sum_{k=0}^{\infty} a_{kn}\, X_k = 0\ ,\ n \geq 1\ , \tag{17}$$

where a change of unknowns has been performed

$$(-1)^k c_k = -(3-4\nu)\, \frac{\pi}{2}\, \varphi_0\, X_k \tag{18}$$

and coefficients are determined as follows

$$\begin{aligned} \widetilde{a}_{k0} = {} & \frac{2(1-\nu)(3-4\nu)}{k+\frac{1}{2}} + \frac{3-4\nu}{k-\frac{3}{2}} + \frac{3}{k+\frac{5}{2}} - \frac{2}{k+\frac{1}{2}}\ , \\ a_{kn} = {} & \frac{2(3-4\nu)}{k-\frac{3}{2}-n} + (n+2)\left[\frac{n+3}{k+\frac{5}{2}+n} - \frac{n+2}{k+\frac{1}{2}+n}\right] \\ & -(n-1)\left[\frac{n+1}{k+\frac{1}{2}+n} - \frac{n}{k-\frac{3}{2}+n}\right] - \frac{(3-4\nu)^2}{k+\frac{1}{2}+n}. \end{aligned} \tag{19}$$

Anticipating a later results, we note that the infinite system of Equations (17) is singular and the substantiation of this fact is cited below.

As follows from the very statement of the problem, the stresses must increase without bound at the corner points $r=\varepsilon$, $\vartheta = \pm\pi$. The pattern of this singularity is known (see [15, pp. 149–153]). Hence we should set up the following asymptotic behaviour of the unknowns $X_k$

$$X_k \sim X_k^{(as)}, \quad k \to \infty, \quad \text{with } X_k^{(as)} = \frac{C_\gamma}{k^{1+\gamma}}, \tag{20}$$

where $0 < \gamma < 1$, $C_\gamma$ is a real constant. The values of $\gamma$ and $C_\gamma$ are to be determined from the subsequent solution.

Allowing for the values of the coefficients $a_{kn}$ in (19) and for exact sums (Profs J. Boersma, N. G. de Bruijn, private communication),

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^{1+\gamma}} \cdot \frac{1}{k+(n+a)} &= -\frac{\pi}{\sin \pi\gamma} \cdot \frac{1}{(n+a)^{1+\gamma}} + \sum_{m=0}^{\infty} \frac{(-1)^m\, \zeta(1+\gamma-m)}{(n+a)^{m+1}}\ , \\ \sum_{k=1}^{\infty} \frac{1}{k^{1+\gamma}} \cdot \frac{1}{k-(n+a)} &= \pi \frac{\cot \pi\gamma - \cot \pi a}{(n+a)^{1+\gamma}} - \sum_{m=0}^{\infty} \frac{\zeta(1+\gamma-m)}{(n+a)^{m+1}}, \end{aligned} \tag{21}$$

where $\zeta(z)$ is the Riemann $\zeta$ function, one may establish the following asymptotic equality

$$C_\gamma \sum_{k=1}^{\infty} \frac{a_{kn}}{k^{1+\gamma}} \sim -\frac{16(1-\nu)^2\,\zeta(1+\gamma)}{n}\,C_\gamma - \frac{4\pi}{\sin\pi\gamma}\cdot\frac{C_\gamma}{n^{1+\gamma}}\left[(3-4\nu)\sin^2\frac{\pi\gamma}{2}+\gamma^2-4(1-\nu)^2\right], \quad n\to\infty \tag{22}$$

It is obvious that the first term on the right-hand side of the equality (22) diminishes logarithmically. This decay is less rapid than required for regularity. Thus, the system of Equations (17) is singular.

### 3.4. INFINITE REGULAR SYSTEM

Putting $X_k = X_k^{(as)}$, $k \geq N$, due to the asymptotic behaviour (20), we transform Equations (17) into the form

$$\sum_{k=0}^{\infty} a_{kn}\,X_k = a_{0n}\,X_0 + \sum_{k=1}^{N-1} a_{kn}\left(X_k - X_k^{(as)}\right) + \sum_{k=1}^{\infty} a_{kn}\,X_k^{(as)}, \quad n \geq 1. \tag{23}$$

Since displacements are to be continuous in the boundary conditions (and they are), the terms on the left-hand side of Equation (17) must not decrease less rapidly than $1/n^2$ when $n \to \infty$. Hence, in view of Equations (22), (23) and the asymptotic behaviour of $a_{kn}$,

$$a_{kn} \sim -\frac{16(1-\nu)^2}{n}, \qquad n \to \infty \quad (\text{for fixed } k), \tag{24}$$

we have

$$X_0 + \sum_{k=1}^{N-1}\left(X_k - X_k^{(as)}\right) = -\zeta(1+\gamma)\,C_\gamma, \tag{25}$$

$$(3-4\nu)\sin^2\frac{\pi\gamma}{2} + \gamma^2 - 4(1-\nu)^2 = 0. \tag{26}$$

When the Poisson ratio $\nu$ is given, one can determine the value of $\gamma$ which is necessary for the asymptotic behaviour (20) from the transcendental Equation (26). As an example, when $\nu = 1/3$, the correct root for $0 < \gamma < 1$ is $\gamma = 0{\cdot}69$.

Thus, the system of Equations (17) becomes regular in terms of expression (20) when the conditions (25) and (26) are satisfied, otherwise the problem is unphysical. Making the substitution of the unknowns as

$$X_k - X_k^{(as)} = Y_k, \qquad 1 \leq k \leq N-1, \tag{27}$$

and rejecting in the system (23) the $N+1, N+2, \ldots$ equations, we obtain the following finite system

$$\begin{aligned} &\widetilde{a}_{00}\,X_0 + \sum_{k=1}^{N-1}\widetilde{a}_{k0}\,Y_k + C_\gamma \sum_{k=1}^{\infty}\frac{\widetilde{a}_{k0}}{k^{1+\gamma}} = 1,\\ &a_{0n}\,X_0 + \sum_{k=1}^{N-1} a_{kn}\,Y_k + C_\gamma \sum_{k=1}^{\infty}\frac{a_{kn}}{k^{1+\gamma}} = 0, \qquad n = 1, 2, \ldots, N \end{aligned} \tag{28}$$

The finite system of Equations (28) involves $N+1$ unknowns, *i.e.*, $X_0, Y_1, \ldots, Y_{N-1}, C_\gamma$ and the same number of equations and it may be used for an approximate solution of the regular infinite system. In fact, this approach to solving the infinite system of Equations (17) is the traditional method of reduction. All numerical results are discussed in Section 4.

### 3.5. Expressions for stresses in the contact zone

Similar to the expressions (9), (10) and (13), the complete solution for the stresses $\sigma_r$, $\tau_{r\vartheta}$ can be written at the surface $r=\varepsilon$ $(t=0)$ as

$$\begin{aligned}\left.\frac{\sigma_r}{2G}\right|_{t=0} &= \sum_{n=1}^{\infty}\Big[(n-1)(n+2)d_{n-1}-(n+1)(n+2)d_{n+1}\Big]\sin n\vartheta \\ &\quad -\sum_{k=0}^{\infty}\Big(k-\frac{1}{2}\Big)c_k\left[\Big(k+\frac{5}{2}\Big)\sin(k+\frac{1}{2})\vartheta-\Big(k-\frac{3}{2}\Big)\sin(k-\frac{3}{2})\vartheta\right], \\ \left.\frac{\tau_{r\vartheta}}{2G}\right|_{t=0} &= 2d_1-\sum_{n=1}^{\infty}\big[n(n-1)d_{n-1}-(n+1)(n+2)d_{n+1}\big]\cos n\vartheta \\ &\quad +\sum_{k=0}^{\infty}\Big(k-\frac{1}{2}\Big)c_k\left[\Big(k+\frac{1}{2}\Big)\cos(k+\frac{1}{2})\vartheta-\Big(k-\frac{3}{2}\Big)\cos(k-\frac{3}{2})\vartheta\right].\end{aligned} \tag{29}$$

By eliminating $d_{n-1}$ and $d_{n+1}$ between the above expressions (29) (see Subsection 3.3), by means of Equations (18), (20) and (27) and the expansions

$$\begin{aligned}\sin\Big(k+\frac{1}{2}\Big)\vartheta &= \frac{(-1)^k}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{k+\frac{1}{2}-n}-\frac{1}{k+\frac{1}{2}+n}\right]\sin n\vartheta, \\ \cos\Big(k+\frac{1}{2}\Big)\vartheta &= \frac{(-1)^k}{\pi}\cdot\frac{1}{k+\frac{1}{2}}+\frac{(-1)^k}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{k+\frac{1}{2}-n}+\frac{1}{k+\frac{1}{2}+n}\right]\cos n\vartheta,\end{aligned}$$

we can write Equations (29) as

$$\begin{aligned}\left.\frac{\sigma_r}{2G}\right|_{t=0} &= \sum_{n=1}^{\infty}(-1)^n\,\sigma_n\cdot\sin n\vartheta, \\ \left.\frac{\tau_{r\vartheta}}{2G}\right|_{t=0} &= \tau_0+\sum_{n=1}^{\infty}(-1)^n\,\tau_n\cdot\cos n\vartheta,\end{aligned} \tag{30}$$

where

$$\begin{aligned}\sigma_n &= -(3-4\nu)\,\varphi_0\left[\alpha_{0n}X_0+\sum_{k=1}^{N-1}\alpha_{kn}Y_k+C_\gamma\sum_{k=1}^{\infty}\frac{\alpha_{kn}}{k^{1+\gamma}}\right],\ n\geq 1, \\ \tau_n &= -2(1-\nu)(3-4\nu)\,\varphi_0\left[\beta_{0n}X_0+\sum_{k=1}^{N-1}\beta_{kn}Y_k+C_\gamma\sum_{k=1}^{\infty}\frac{\beta_{kn}}{k^{1+\gamma}}\right],\ n\geq 1, \\ \tau_0 &= -\varphi_0\left[\widetilde{\beta}_{00}X_0+\sum_{k=1}^{N-1}\widetilde{\beta}_{k0}Y_k+C_\gamma\sum_{k=1}^{\infty}\frac{\widetilde{\beta}_{k0}}{k^{1+\gamma}}\right],\end{aligned} \tag{31}$$

with the coefficients

$$\begin{aligned}
\alpha_{kn} = {} & (1-2\nu)\frac{n+1}{k+\frac{1}{2}+n} - (n+1)\left(\frac{n-1}{k+\frac{1}{2}-n} - \frac{n}{k-\frac{3}{2}-n}\right) \\
& + \frac{2(1-\nu)}{3-4\nu}(n-1)\left(\frac{n+1}{k+\frac{1}{2}+n} - \frac{n}{k-\frac{3}{2}+n}\right), \\
\beta_{kn} = {} & \frac{n+1}{k+\frac{1}{2}+n} + \frac{n-1}{3-4\nu}\left(\frac{n+1}{k+\frac{1}{2}+n} - \frac{n}{k-\frac{3}{2}+n}\right), \\
\widetilde{\beta}_{k0} = {} & \frac{3-4\nu}{k-\frac{3}{2}} + \frac{3}{k+\frac{5}{2}} - \frac{2}{k+\frac{1}{2}}\,.
\end{aligned} \tag{32}$$

The conditions (25) and (27) permit us to establish the following asymptotic behaviour for the coefficients of the series (30)

$$\sigma_n \sim \frac{S_\gamma}{n^\gamma}\,, \quad \tau_n \sim \frac{T_\gamma}{n^\gamma}\,, \qquad n \to \infty\,, \tag{33}$$

with the notations

$$\begin{aligned}
S_\gamma &= \frac{\pi}{\sin\pi\gamma}\cdot C_\gamma\,\varphi_0\Big[(1-2\nu)(3-4\nu) + (1+2\gamma)\Big((3-4\nu)\cos\pi\gamma - 2(1-\nu)\Big)\Big], \\
T_\gamma &= 2(1-\nu)\frac{\pi}{\sin\pi\gamma}\cdot C_\gamma\,\varphi_0\Big[(3-4\nu) - (1+2\gamma)\Big].
\end{aligned} \tag{34}$$

As emphasized above, the normal stresses as well as that representing shearing increase (or decrease) without bound near the corner points of the contact surface between the rigid shaft and the elastic plane, *i.e.*, at $r = \varepsilon$, $\vartheta = \pm\pi$. To estimate the asymptotic behaviour of the stresses in these singular points, we use the following asymptotic equalities[2] when $\vartheta \to \pm\pi$

$$\begin{aligned}
\sum_{n=N}^{\infty}\frac{(-1)^n}{n^\gamma}\cos n\vartheta &\sim \frac{\pi^2}{(2\pi)^\gamma\,\Gamma(\gamma)\cos\frac{\pi\gamma}{2}}\cdot\frac{1}{\left(\pi^2-\vartheta^2\right)^{1-\gamma}}\,, \\
\sum_{n=N}^{\infty}\frac{(-1)^n}{n^\gamma}\sin n\vartheta &\sim \mp\frac{\pi^2}{(2\pi)^\gamma\,\Gamma(\gamma)\sin\frac{\pi\gamma}{2}}\cdot\frac{1}{\left(\pi^2-\vartheta^2\right)^{1-\gamma}}\,.
\end{aligned} \tag{35}$$

Finally, with the aid of Equations (30) and the asymptotic expressions (33), (35) we obtain the asymptotic expansions of $\sigma_r$ and $\tau_{r\vartheta}$ at the surface $r = \varepsilon$ for $\vartheta \to \pm\pi$

$$\begin{aligned}
\left.\frac{\sigma_r}{2G}\right|_{\vartheta\to\pm\pi} &\sim \mp\left(\frac{\pi}{2}\right)^\gamma\frac{S_\gamma}{\Gamma(\gamma)\sin\frac{\pi\gamma}{2}}\cdot\frac{1}{\left(1-\frac{\vartheta^2}{\pi^2}\right)^{1-\gamma}}\,, \\
\left.\frac{\tau_{r\vartheta}}{2G}\right|_{\vartheta\to\pm\pi} &\sim \left(\frac{\pi}{2}\right)^\gamma\frac{T_\gamma}{\Gamma(\gamma)\cos\frac{\pi\gamma}{2}}\cdot\frac{1}{\left(1-\frac{\vartheta^2}{\pi^2}\right)^{1-\gamma}}\,,
\end{aligned} \tag{36}$$

[2] To derive these relations, one should use the Fourier expansions of the functions $1/\left(\pi^2-\vartheta^2\right)^{1-\gamma}$ and $\vartheta/\left(\pi^2-\vartheta^2\right)^{1-\gamma}$ in terms of $\sin n\vartheta$, $\cos n\vartheta$ on the interval $[-\pi,\pi]$.

where $S_\gamma$ and $T_\gamma$ are defined by Equations (34).

In conclusion, some remarks concerning the stress behaviour at the points $r = \varepsilon$, $\vartheta = \pm\pi$ should be made. First of all the order of the singularity is to be distinguished. The stress field diverges as $1/(1 \mp \vartheta/\pi)^{1-\gamma}$ when $\vartheta \to \pm\pi$; it depends on the elasticity parameters of the medium. Contrary to fracture mechanics, this is not the inverse-square-root singularity at the crack tip.

Moreover, there is a certain inconsistency in the shear-stress behaviour at the location $r = \varepsilon$, $\vartheta = \pm\pi$. It depends on the path by which these points are approached – from the side of the elastic medium or from the traction-free slit region. The stresses are singular in the former case and must be zero due to (4) in the latter. This character of the stress field is inevitable. It occurs in a number of other problems, including bonded contact between elastic bodies and rigid boundaries. The introduction of singular points, in which characteristics of the stress field are uncertain and fundamentally different in their close proximity, promotes field analysis (see Grinchenko and Ulitko [19]).

## 4. Numerical results

The results of our calculations of the stress distribution at the surface $r = \varepsilon$ are presented in Figures 2 and 3. To plot these graphs we used the expressions (30–32). The values of $X_0$, $Y_1, \ldots, Y_{N-1}$, $C_\gamma$ were evaluated by solving the reduced infinite system (28). Here the cases $N = 30$ and $N = 100$ are illustrated when Poisson's ratio is $\nu = 1/3$. The infinite sums in Equations (28), as well as in the expressions (30), (31), were calculated with a fixed accuracy. The value of $C_\gamma$ turns out to be sensitive to changing the number of equations in the reduced system (28).

Note, that the pattern of stress distributions for other values of Poisson's ratio is analogous to Figures 2 and 3. Moreover, it is not essential what exact value of $N$ we choose and these illustrations are sufficient for our references[3].

Figures 2 and 3 represent the distributions of stresses for $N = 30$ and $N = 100$, respectively. In Figures 2(a) and 3(a) the normal stress $\sigma_r$ as well as that for shearing $\tau_{r\vartheta}$ are drawn on the large scale of the vertical axis. In Figures 2(b) and 3(b) only $\tau_{r\vartheta}$ is scaled down. Remembering that the antisymmetric case is considered, we have plotted the stress distributions on the interval $0° \le \vartheta \le 180°$. That is, in order to obtain the stress field for $\vartheta$ in the range $-180° \le \vartheta \le 0°$ one should map the $\sigma_r$- and $\tau_{r\vartheta}$-curves symmetrically about the origin and about the vertical axis, respectively.

Moreover, it is relevant for engineering purposes to compare torsional stiffnesses of the rigid disk inserted into the slitted infinite plane and the same inserted into an intact plane. This way one can assess the influence of the slit on the torsional stiffness. Thus, in view of Equation (5), on the basis of calculations for $\nu = 1/3$ and $N = 100$, the following relation was determined: $M_0 = 12{\cdot}27\, G\varphi_0\varepsilon^2$. For the intact plane it is of the form $M = 4\pi\, G\varphi_0\varepsilon^2$. Hence, in the case with the slit the torsional stiffness of the disk slightly decreases in comparison with the intact plane. This inference was predicted intuitively.

[3]It should be mentioned that, on the basis of numerical techniques, we designed a program which allows to solve the system (28) for arbitrary values of $\nu$ and $N$.

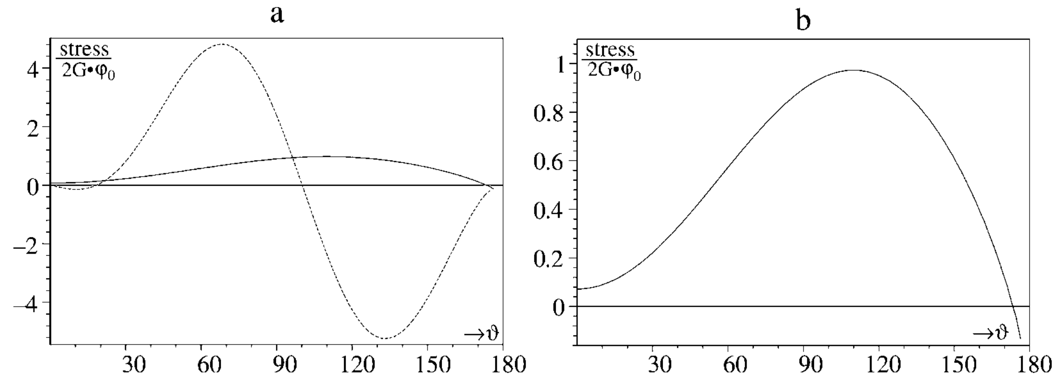


*Figure 2.* Stress distribution along the force-transfer surface calculated for $N = 30$: (a) dashed and solid lines represent normal and shearing stresses, respectively; (b) shearing stress scaled down.

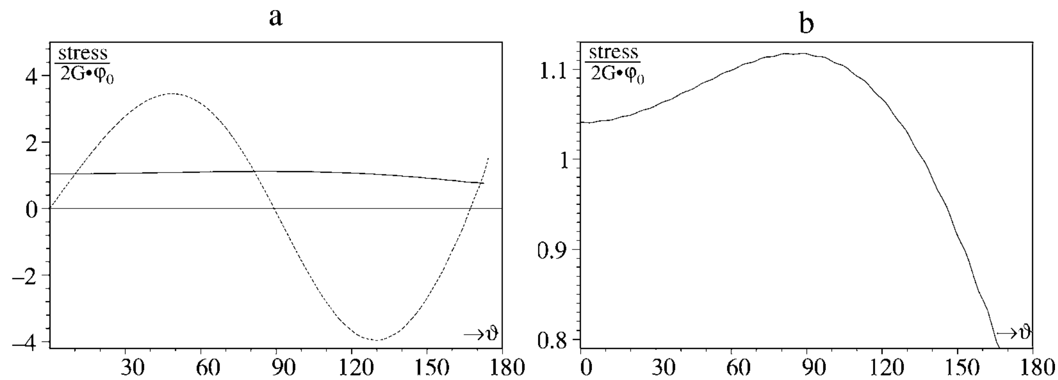


*Figure 3.* Stress distribution along the force-transfer surface calculated for $N = 100$: (a) dashed and solid lines represent normal and shearing stresses, respectively; (b) shearing stress scaled down.

## 5. Discussion

Let us recall the solution obtained by Neuber [13] for the slitted plane

$$\begin{aligned}\sigma_r &= \frac{M}{2\pi \cdot r^2}\left\{-2\sin 2\vartheta + B\sqrt{\frac{r}{\varepsilon}}\left[\sin\frac{\vartheta}{2} - 7\sin\frac{3\vartheta}{2}\right]\right\},\\ \sigma_\vartheta &= \frac{M}{2\pi \cdot r^2}\left\{B\sqrt{\frac{r}{\varepsilon}}\left[-\sin\frac{\vartheta}{2} - \sin\frac{3\vartheta}{2}\right]\right\},\\ \tau_{r\vartheta} &= \frac{M}{2\pi \cdot r^2}\left\{\cos 2\vartheta - 1 + B\sqrt{\frac{r}{\varepsilon}}\left[\cos\frac{\vartheta}{2} + 3\cos\frac{3\vartheta}{2}\right]\right\},\end{aligned} \tag{37}$$

where $B = -\sqrt{3}/9$. See Figure 4 for the corresponding stress distribution. This solution contains only two components. They are homogeneous solutions and the former agrees with the Carothers solution for $\alpha = \pi$, while the latter possesses the self-equilibrated peculiarity about the origin. The solution (29) involves analogous terms. They contain, respectively, the coefficients $d_1$ and $c_1$ (see Subsection 3.1).

Among all the homogeneous solutions Neuber selected only two. Why should one take into account not three, four, … homogeneous solutions, but just these two? In this paper we checked this approach for a specific case.

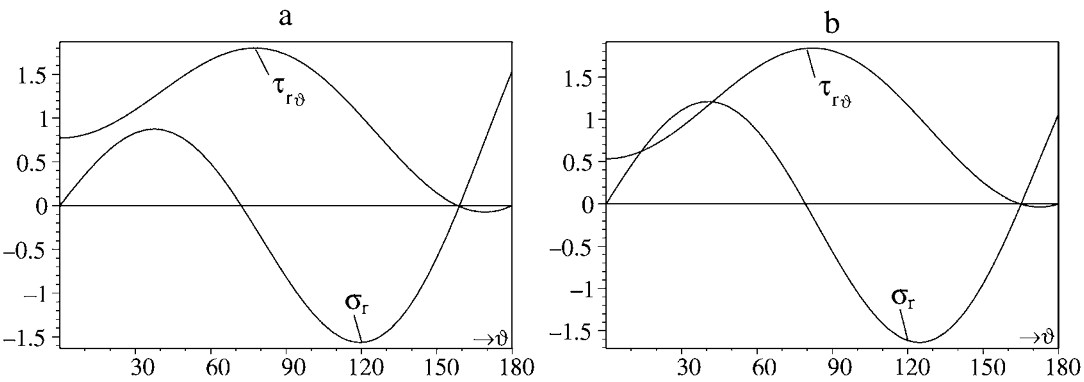


*Figure 4.* Stress distribution along the force-transfer surface for the slitted elastic plane: (a) according to Neuber [13]; (b) according to the energy method after Ulitko [14], where $B = -0{\cdot}133$.

Comparing the pictures in Figures 2 and 3 with corresponding results by Neuber [13] (see Figure 4), we may conclude that they are in complete qualitative agreement. It refers in particular to the behaviour of shearing stress. Its distribution fully complies with the principles of force transfer after Neuber [13] and it is actually smoother than in Figure 4. Moreover, one can reveal the trend towards fairing the curve of $\tau_{r\vartheta}$ due to an increase of $N$. This tendency remains when $N$ is larger than 100. However, even for $N > 100$, no prominent variations of the stress distribution are noticeable.

Thus, Neuber's results based on his principles of force transfer are not at variance with ours. They would be invaluable if they were backed up by minimum principles, that is, if addition of every next homogeneous solution to the solution (37) would cause the strain energy of the truncated wedge to increase. But further calculations contradict this expectation.

## 6. Conclusions

Neuber [13] demonstrated the method of constructing the practically applicable solution of the wedge problem only by means of two roots of the eigenequation (12), despite the failure of the Saint-Venant principle. Equations like (12) are thoroughly examined in the work [15]. First of all, they have infinitely many roots, which are complex in general. Due to minimum principles of elasticity, one may take account of those, which render the potential energy of the elastic domain finite. The energy method is based on these principles too, but it again employs two homogeneous solutions selected by Neuber [13].

The proposed method of homogeneous solutions is a powerful tool for solving many problems of elasticity for bounded elastic domains, as in the problem of thick plates (see also Lourje [16]). In order to realize it, we accounted for all the roots of Equation (12) and corresponding homogeneous solutions which satisfy the minimum principles. Therefore, our solution includes every possible self-equilibrating systems and hence we called it the exact solution by right. Despite the fact that most of the transformations are cumbersome and tedious, the final results appear rather transparent for further numerical and analytical simulations. In particular, the distributions of stresses along the force-transfer surface were obtained by numerical calculations based on the reduction method. These graphs confirmed the correctness of Neuber's approach which is elementary enough and requires no special explorations.

In conclusion, it should be noted that, unfortunately, the technique of reducing the boundary-value problem to the infinite system of linear algebraic equations suggested in this paper appears useless for other wedge angles, since it is impossible to convert one subset of homogeneous solutions with respect to another. But we suppose that for apex angles in the range $2\alpha^* \leq 2\alpha \leq 2\pi$ the state of things will be similar to the slitted elastic plane.

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